How WEDGE You Teach the Unit-ANGLE Concept?

By Gayle M. Millsaps

Fifth graders rethink what calculating angle means.

The concepts of angle and angle measure have been acknowledged as difficult for elementary school students to grasp (Strutchens, Martin, and Kenney 2003). The Wedge activity (Browning and Garza-Kling 2009; Van de Walle 2004; Wilson 1990) can provide an opportunity for students to examine their understanding of angle measurement and to rethink what it means to measure angles. The activity can be used to support a learning trajectory for students' development of the angle-measurement concept. Teachers who can recognize where their students are on a learning trajectory for a topic can make effective decisions about how to move students along the trajectory to the ultimate learning goal.

The following narrative illustrates the learning trajectory for students' development of angle-measurement concepts within the context of a fifth-grade classroom. Students first learn to differentiate obtuse and acute angles (Lehrer, Jenkins, and Osana 1998). When they begin to identify right and straight angles, they fairly quickly integrate these special angles into their prior knowledge as benchmarks of angle size. The next step in a learning trajectory for angle measurement is ordering angles by size, followed by the development of a simple unit-angle concept, and culminating in the abstraction of the unit angle.

The setting

My colleague, Mrs. Wise, teaches fifth grade and is well versed in traditional methods of instruction about protractors and angle measurement. As a participant in the Making Algebra Realistic Knowledge (MARK) Project, she has been introduced to an inquiry approach to mathematics instruction. Her experiences implementing an inquiry approach to teaching number concepts has led her to recognize that an inquiry approach can encourage students to develop richer and more connected mathematical concepts. She asked another colleague and me for help in developing an inquiry lesson on the topic of angle measurement.

Wise had a class of academically successful students, most of whom were competent using the protractor after her instruction, but she realized they had only a superficial concept of angle measurement. She reflects:

> When my students were handed the protractor for the first time in my class, they didn't know how to lay it on an angle. They didn't know what all the lines meant on the protractor. They certainly didn't get what a unit (degree) meant. Students coming into fifth grade do not typically measure angles before coming to me. They may recognize an angle and [may] have memorized what *acute*, *obtuse*, and *right* mean (their definitions), but they don't know what the units are to make an angle qualify as acute, obtuse, and right.

Many teachers who are considering reform-minded approaches to instruction share Wise's dilemma and often ask, "Can I teach the skills and then go back and teach the concepts underlying those skills?" Battista and Larson (1994) argue against such an approach. Traditional instruction is not recommended before inquiry instruction on a topic because focusing on the execution of procedures and attention to

superficial characteristics of mathematics problems can interfere with students' development of important mathematical concepts. Nevertheless, teachers may find themselves needing to address students' understanding of a topic after the students have been exposed to the topic through traditional instruction. For this reason, I suggested a lesson I use with preservice elementary school teachers in which students create an angle-measure tool—that is, the Wedge activity. Preservice elementary school teachers have all had prior experiences with using protractors, often without understanding the unit angle. To allow my students to revisit what it means to measure an angle, I provide a contextual setting in which recreating a tool for measuring angles is meaningful: an archeological expedition. Wise revised the context for her classroom to that of a tribe in the rainforest.

Wise invited us to observe and participate in the lesson she had designed based on our suggestions. Her lesson and the reflection on the lesson illustrate both an inquiry approach to instruction and a learning trajectory for angle-measurement concepts. The discussion of how the lesson reveals the learning trajectory is interwoven in the description of the lesson.

The lesson

Wise began by having students use compasses to construct circles. When the students asked if the circles should be a certain size, she told them they could be any size. Because her students were studying the rainforest, Wise chose a related contextual setting for the activity. In Wise's story, a tribe in the rainforest did not have protractors. She asked her students to (1) discuss in their small groups how the tribe might use the circles they had created to measure angles and (2) report what they had discussed. Students' reports revealed some of their current ideas about measuring angles:

- **"If** you took something straight, you could draw a straight line up, and then you could try to make it go 90 degrees."
- **"You** could put a mark on your circle to say whether it was acute, right, or obtuse."
- **"The** whole circle is 360 degrees; so if you cut it in half, that straight line would be 180 degrees. If you cut it into fourths, … each angle would be 90 degrees."

Two things likely contributed to students' initial ideas for creating an angle measurement tool: (1) their prior experience using protractors and (2) their understanding relative to the learning trajectory for angle measurement. Wise's reference to protractors surely prompted students to think of 90 degrees, 180 degrees, and 360 degrees as they attempted to create a tool for measuring angles. However, evidence exists that students' current understanding of angle measurement also played a part.

It is not surprising that students' first ideas for creating a tool for measuring angles would indicate the initial steps in an angle-measurement learning trajectory. They clearly knew the benchmark angles and could differentiate relative sizes of angles using the terms *acute*, *obtuse*, and *right*—that is, they could make angle-size comparisons. However, this ability does not necessarily imply that knowing the term *degrees* means that students have the concept of degrees as unit angles. *Degrees* may mean something else to the students, such as a label that is attached to a number, where the number represents a relative location and not a number of units. Thus, at the angle-size comparison step in an angle-measurement trajectory, angle sizes are derived qualitatively through direct comparisons or comparisons with internalized benchmarks and not quantitatively through counting numbers of units.

To assess whether some students had developed a simple unit-angle concept, Wise extended the scenario. Students were shown how fictitious tribespeople in the rainforest might fold their circles by halving again and again (see **fig. 1**). Wise asked students to again discuss in their groups how the invented tribe might use the folded circles to measure angles if they do not know *degrees*. Despite this extension of the scenario, students continued to relate the folded circles to their prior experience of measuring angles with protractors using degree measure. For example, one student shared:

> I folded my circle until I folded it in half, which would be 180 degrees. Then I folded it another half, which would be 90 degrees. Then I folded it one more time, and you'd get 35, 45 degrees. So if they wanted to draw a small angle and it's, say 45, they would just draw over from one crease to another crease.

FIGURE 1

To assess whether her students had developed a simple unit-angle concept, Wise asked them how fictitious tribespeople might have used folded circles to measure angles if they did not know the term *degrees*.

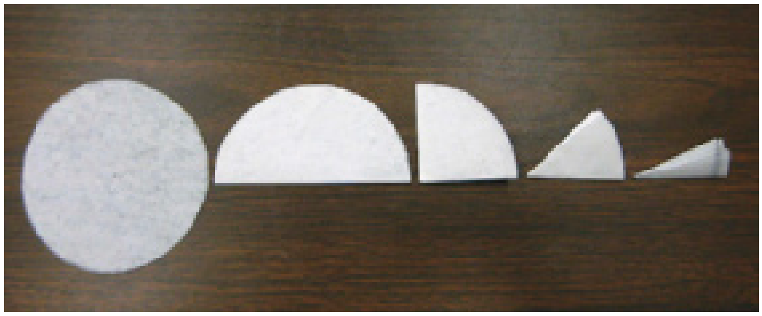
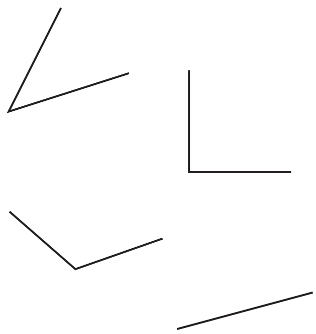


FIGURE 2

Despite Wise's limits on vocabulary, some students continued to relate the circle tool to their previous experiences of measuring angles with protractors.



When the teacher asked a different student, "Why did you stop there?" (at 45 degrees), he responded, "Because I can't divide." He related the circle to his experiences with locations on a protractor. He correlated the parts of the circle with a number of degrees. He identified half of a circle with 180 and reasoned that half of a half-circle would be 90 degrees and so on. However, the relationship with the protractor limited his production of smaller potential angle units.

Despite the possibility that some students may not have understood that they were creating a unit angle, Wise asked students to think of a name for the unit of measure that had been created by folding their circles. She reinforced the point that fictitious tribespeople would not call the unit *degrees*. Although some small groups discussed names that could refer to a unit amount, such as *quantonimoes* and *fractions, like eighths*, half the small groups offered such terms as *acute* and *obtuse*, which refer to angle-size comparisons. They used the terms *large*, *medium*, and *small* as substitutes for the terms *obtuse*, *right*, and *acute*. This is evident in the following student explanation:

> Medium, small, large, like from the top, which would be the 90-degree angle, because there would be a half circle, which would be 180. If it went to the very top, it would be the 90-degree angle, and we could label that *medium*. And if it was on the right, it would be *small*; and if it was on the left, it would be *large*.

Wise next asked students to discuss how they would use the "wedges" to measure angles. One student observed,

> I think, to measure anything with this new protractor, we would have to punch a hole in the middle, because we need to see the vertex.

The use of benchmark (right and straight) angles and angle-comparison terms (*acute* and *obtuse*) continued to be evident in students' thinking about the circle tool. For example, a student said,

> So, what if you just folded it? I am going to call these *trunks*. If you fold it in half, it is one trunk; if you fold it again, it is a smaller trunk. So that would be about a right angle. If you fold this again, it would be acute; if you fold it again, it would be a "cuter" acute angle.

Yet another student's explanation showed some evidence of the use of a simple unit angle:

> If you do one wedge, it would be a small angle. If you have two wedges, it is a right angle. If you have three wedges, you have an obtuse angle, or big.

Wise gave her students a sheet of paper with four angles to measure (see **fig. 2**) with their circle tools and the instructions, "You can't use *obtuse* or *right* or *acute* right now; how are you going to measure these angles with your new protractor, and what are you going to call your

answer?" Using the circle tool, students worked in small groups to measure the angles.

In her orchestration of the whole-class discussion, Wise began with a group that used only comparisons to benchmark angles to indicate the relative sizes of the angles as they used the circle tool. Brian reported for the group while pointing to various parts of his folded circle under the document camera. From the lack of clarity in Brian's report, we can conjecture that he has not developed a concept of unit angle:

> OK. So we figured out that this meets this. So basically this is this, so we called it *medium*, er, *small half*. *Half* would be this [*the right angle on* **fig. 2**], so this would be *half of a*, uh, it would be—the small half is *half*, basically half of this [quarter circle]. So *half* would be half of half the circle. Then if you take a *large half*, it is anything bigger than *medium* is *large* [*referring to the obtuse angle on* **fig. 2**]. So, then if you took the half circle, that would be half of the whole circle [*referring to the straight angle on* **fig. 2**]. So, if you said *half*, it is half of the circle. If you said *large half*, it is bigger than the medium half; and if you said *medium half*, it would be half of the half circle. So, say there was four of these [quarter circles] that we put together in a circle, and two of these [half circles] together would be a circle. Then this [*pointing to an eighth circle*] is a small half, so that is half of this [quarter circle], basically.

In the subsequent discussion, students from other groups asked Brian to explain: "What is a large?" and "What is a half?" Brian's answers to their questions indicate that he has associated parts of a circle with benchmark angles: the whole circle, the *half* or *half of a circle* (a straight angle or a 180-degree angle), and the *medium half*—or half of a half—of a circle (a right angle or a 90-degree angle). He then categorized other angles by comparing their size to the benchmarks. For him, a *large half* described any obtuse angle, because "anything bigger than medium is large." Whether Brian would similarly label any acute angle as a *small half* is unclear because the acute angle on the handout was a 45-degree angle and thus half of a right angle. Classifying his understanding of angle measure as comparison-based is reasonable. He is judging angle sizes qualitatively; he is not counting angle units.

The next group described a refined scheme that supplied an intermediate step between comparing benchmark angles to indicate the relative sizes of angles and using the wedge as a unit to measure angles. Traci reported:

> We did, like, the first one [*pointing to various parts of* **fig. 3**] would be smallest; the second one would be smaller; the third one would be small medium; and then the 90-degree angle would be medium. Then the next one would be large medium, and the next would be large, and the next would be larger, and the last would be largest. So, if you hold this [*see* **fig. 3**], then two of them, and that would be *small* because it's the second wedge; and the next one [*referring to the right angle in* **fig. 2**] would be *medium* because it's in the middle; and the next one [*referring to the obtuse angle in* **fig. 2**)] would be *large* because there's six wedges; and that one [*pointing to the straight angle in* **fig. 2**] would be largest because that's all of them.

Questions followed Traci's report. Teddy began:

> When you measured it, you counted the distance between the, like how many wedges were there? Couldn't you just look at the lines where it sat on? … Like when you were doing this down here, you said that the largest counted eight in between, but couldn't you just look at the word *largest* and mark it?

Traci responded, "It is just filling up all these spaces."

The second group ordered the angles by size qualitatively, but they also used the wedge as a unit to describe the different angles' measures. Thus, they have moved beyond the qualitative step in the angle-measurement trajectory of comparing angle sizes to the development of a simple unit-angle concept. They can use the wedge as a unit to be counted.

The last group to report actually used the wedge as a unit of measure. Joe reported, "I tried the first wedge and measured the first angle [*see* **fig. 4**], which it didn't do it; but when I opened it up to two wedges, it matched."

When his teacher asked, "So that angle measured what?" Joe explained:

> Two wedges. And then I used two wedges up there, but this [*pointing to the right angle on* **fig. 2**] is bigger than two wedges. I opened it up then to four wedges, and it worked. So, four wedges. And since that [*pointing to the obtuse angle on* **fig. 2**] was even bigger, I opened it up all the way, and I counted the wedges, and it was six wedges around. Then this one [*the straight angle on* **fig. 2**] was obviously a straight line, so I added more units, and that is eight.

Wise again prompted, "So your unit was what to measure?"

Joe answered, "Wedges."

The last group used the "wedge" as a unit angle to measure the angles. They did not indicate a need to compare the angles qualitatively. They understood that counting the number of wedges communicates the size of the angle without referring back to degrees or to the benchmark angles.

The understanding that the last group demonstrated, however, may have still been at the level of a simple unit-angle concept. Group members were able to treat a wedge as a unit to be counted. However, as is evident in the following discussion that Wise monitored between Teddy and Lisa, students had yet to realize that other unit angles may be defined and that one unit angle may be defined in terms of another unit angle. Teddy began, "I think it's good. But he was saying two wedges to make the first one. But if someone folded it one more time, it might be four wedges and they would think that two wedges might just be—"

Wise asked, "What do you think, Lisa?"

"I agree with Teddy because we did it that way too, but we only had eight wedges, and so I had a different answer on mine."

These students recognized that the circle tool could have different sizes of "wedges," and that as a result, the measures they reported would be different. As Wise encouraged the class to try to resolve the problem, students made several conjectures about why differences may occur when "wedges" are used as units. Heather raised the concern that different sizes of circles may have contributed to the different answers. Teddy responded to Heather:

> It wouldn't matter the size of the circle. We could put it like a fraction. Like two wedges over eight wedges, so the bottom would be the total [of] how many wedges.

Teddy's response indicates that he understood that the ratio of the number of wedges that match the angle to the number of wedges

FIGURE 3

Traci's group described a refined scheme that supplied an intermediate step between comparing benchmark angles to indicate the relative sizes of angles and using the wedge as a unit to measure angles.

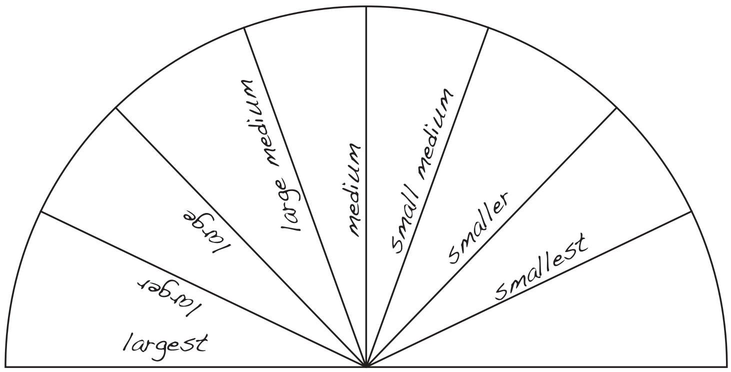
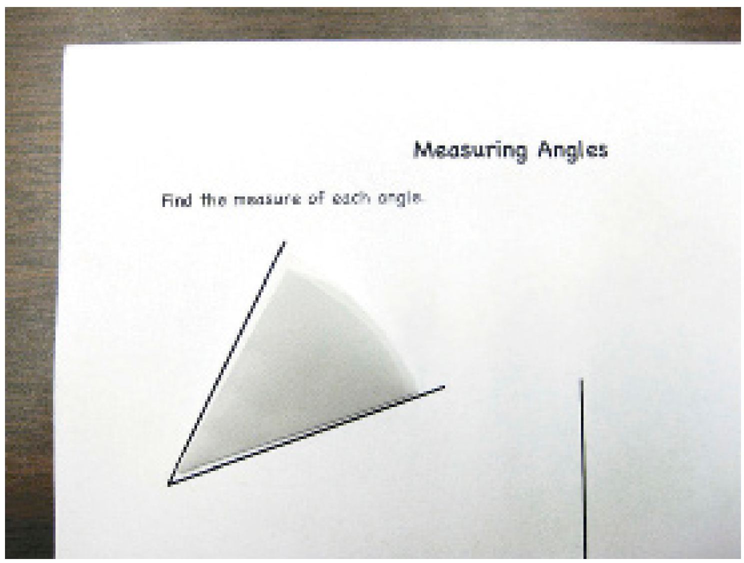


FIGURE 4

Joe's group members used the wedge as a unit angle to measure angles. They understood that counting the number of wedges is sufficient to compare angle size.



in a circle would be the same no matter the size of the circle. Many students, like Ken and Frank, realized that the number of folds of the circle created the difference in the answers. Frank suggested different names for different sizes of wedges, "You could call it a *two-fold wedge* or a *three-fold wedge*." Frank's response indicates his realization that different sizes of units should have different names. There is also evidence that some students observed an equivalence relationship between wedge sizes because halving larger wedges created smaller wedges. Because of time constraints, Wise was unable to fully explore students' suggestions for addressing the problem of different answers when using wedges to measure angles.

Reflection on the lesson

Although not dealt with in the lesson that we observed, the final class discussion suggests that some students in the class were ready to develop an abstract concept of unit angle. Students already recognized that different unit angles could be defined and that the radius of the circle does not contribute to defining a unit angle. More discussion may have confirmed their conjecture that equivalences could be defined between different unit angles and that the definition of a unit angle requires identifying the number of units in a circle. Wise could have capitalized on Heather's conjecture that "the circles should all be the same size," thus, the concepts undergirding unit angle and angle measurement could be explored and confirmed. For example, the teacher could ask Teddy to clarify his response to Heather that if the number of wedges in two circles is the same, then the measures will be the same. Teddy or other students might demonstrate their reasoning by measuring the same angle with different sizes of circles that have the same number of wedges. If not, the teacher could ask, "What if the circles are different sizes?" and require students to explain why circle size does not make a difference.

Ken and Frank's conjecture that the number of wedges in the circle defines the unit angle could also occur in a conversation about whether circle size is important to angle measurement. Some students might argue that same-size circles with different numbers of wedges give different measures for an angle. If not, the teacher could ask students, "What if the circles are the same size but have different numbers of wedges?" A discussion of the relationship between the measures generated by circles with different numbers of wedges could provide an opportunity for students to recognize that different angle units can exist and that equivalences may exist between the different angle units. If students are allowed to use fraction language to describe their wedges, they may exploit connections between equivalent angle units and equivalent fraction units, such as, "One 1/8 wedge is the same size angle as two 1/16 wedges." Once students have become comfortable with the concept of different-size angle units and equivalences between different-size angle units, the teacher could introduce the protractor and ask students, "Can we use what we know about wedges to understand what degrees are? Can we tell how many degrees a one-eighth wedge would equal?" As students reach the point that they can recognize different angle units and the equivalencies between the different angles units, they will have developed an abstract unit-angle concept.

Discussions such as these might offer students like Heather a chance to recognize and define unit angles for themselves as they try to make sense of their peers' reasoning. More important, such discussions give all students the chance to realize that unit size is an important aspect of measurement, that a unit for measuring angles is an angle, and that there is a connection between an angle-measure unit and a fraction of a circle.

Conclusion

Wise's experience affirmed the value of beginning angle-measurement instruction through an inquiry approach using the Wedge activity rather than a traditional approach with the protractor. She has changed her teaching practice. The Wedge activity furnished an opportunity for Wise and her students to examine their understanding of angle measurement and to rethink what it means to measure angles. The activity pushed the development of the concept of unit angle as students discussed with one another how to design and use an angle-measurement tool. The narrative of her lesson provides mathematics teachers with a model for an inquiry approach to instruction and an illustration of the learning trajectory for angle-measurement concepts.

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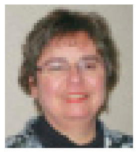

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